Topic 4 – Random Variables, Expected Value, Games

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<u>Def:</u> Let (S, Ω, P) be a probability space. A <u>random variable</u> is a function

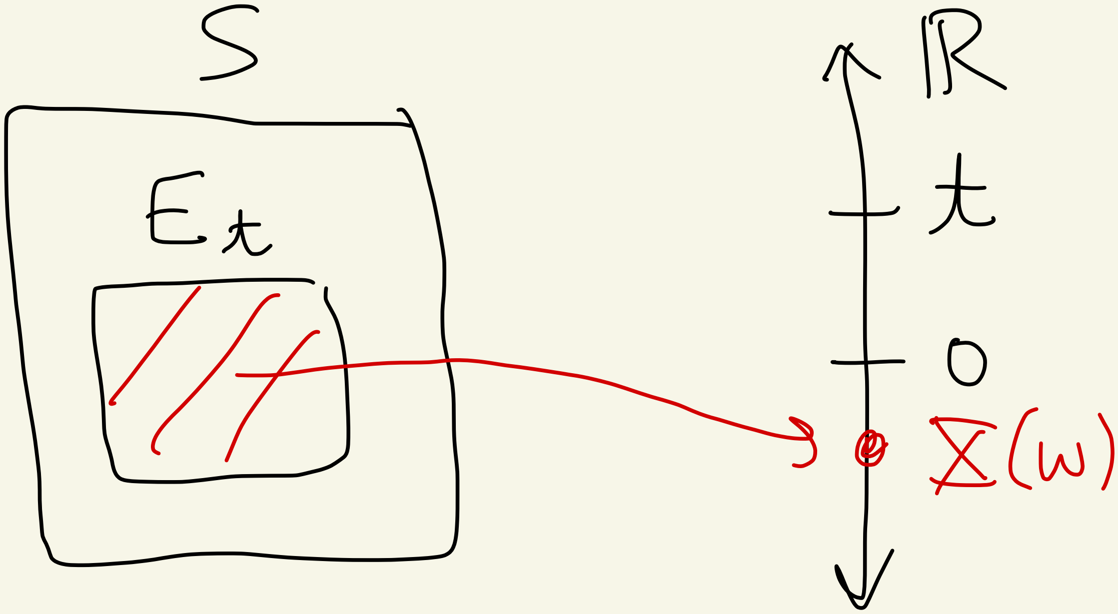
$\underbrace{X : S \longrightarrow \mathbb{R}}$ such that for all real numbers t

X is a function
input $= S$
output $=$ real #s $= \mathbb{R}$

we have that

$$E_t = \{ w \mid w \in S \text{ and } X(w) \leq t \}$$

is an event in Ω.



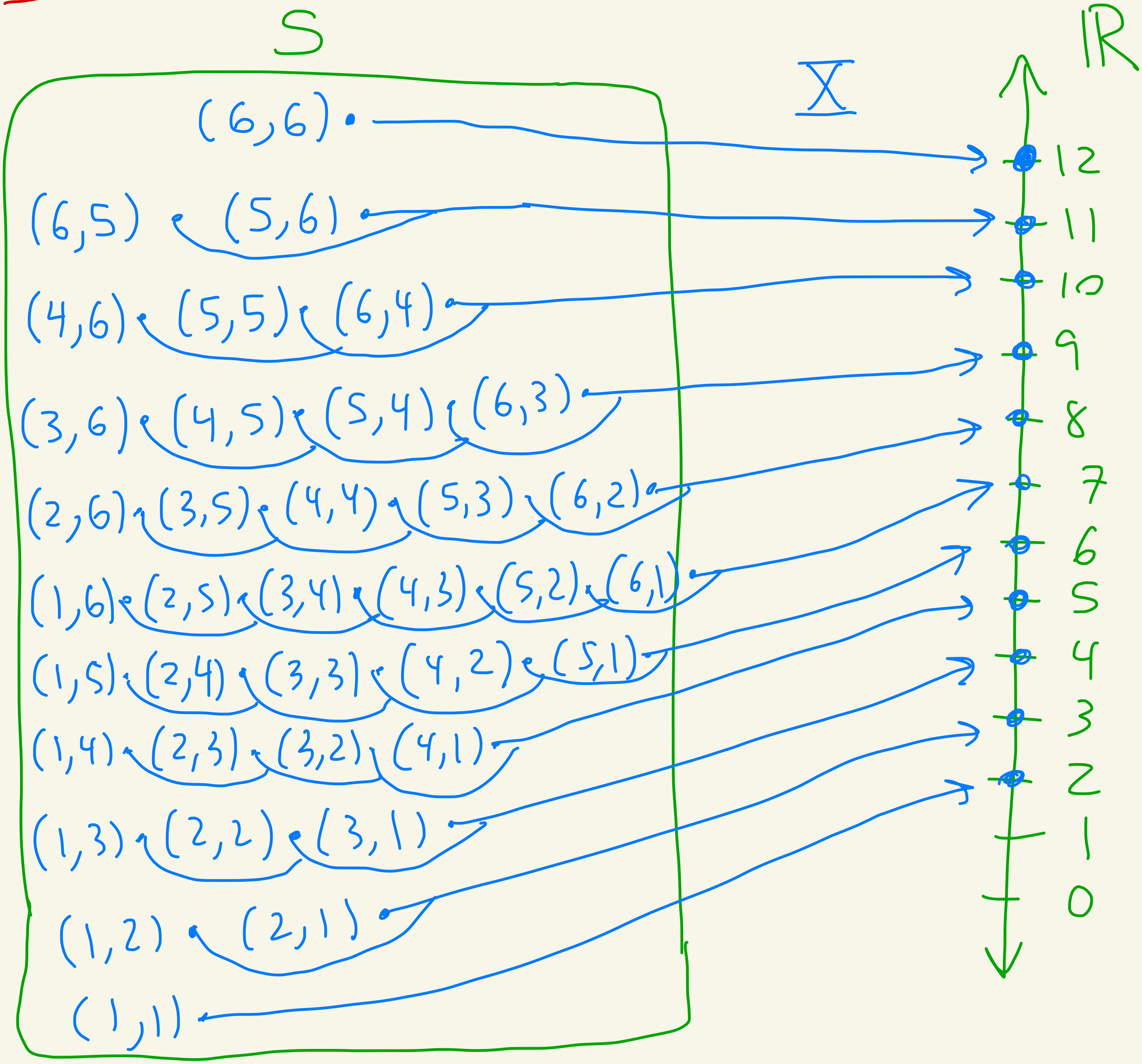
Note: The condition on E_t means we can calculate $P(E_t)$. In our class when S is finite and Ω is all subsets of S this condition will always occur so a random variable is just a function $X : S \to \mathbb{R}$

Def: Let X be a random variable on a probability space (S, Ω, P). We say that X is discrete if the range of X can be enumerated as a list of values $x_1, x_2, x_3, \ldots$

$\boxed{\text{For Math 3450: Ie, the range of } X \text{ is finite or countably infinite}}$

Ex: Let (S, Ω, P) be a probability space corresponding to rolling two 6-sided dice. Let X be the sum of the dice.

For example, $X(2,5) = 2 + 5 = 7$



X is a discrete random variable.
It's range is 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

<u>Def:</u> Let X be a random variable on a probability space (S, Ω, P).

Define:

- $$P(X=i) = P(\underbrace{\{w \mid w \in S \text{ and } X(w)=i\}}_{\text{set of all outcomes } w \text{ where } X(w)=i})$$

- $$P(X \leq i) = P(\underbrace{\{w \mid w \in S \text{ and } X(w) \leq i\}}_{\text{set of all outcomes } w \text{ where } X(w) \leq i})$$

Similar defs can be made for
$P(X<i)$, $P(X \geq i)$, etc.

- The <u>probability function</u> p of X is
$p(i) = P(X=i)$
So, $p: \mathbb{R} \to \mathbb{R}$

- The <u>cumulative distribution function</u> of X is

$$F(i) = P(X \leq i)$$

So, $F : \mathbb{R} \to \mathbb{R}$

<u>Ex:</u> Consider the previous example where (S, Ω, P) represents rolling two 6-sided dice and X is the sum of the dice.

Let's draw the probability function $p : \mathbb{R} \to \mathbb{R}$ and cumulative distribution function $F : \mathbb{R} \to \mathbb{R}$

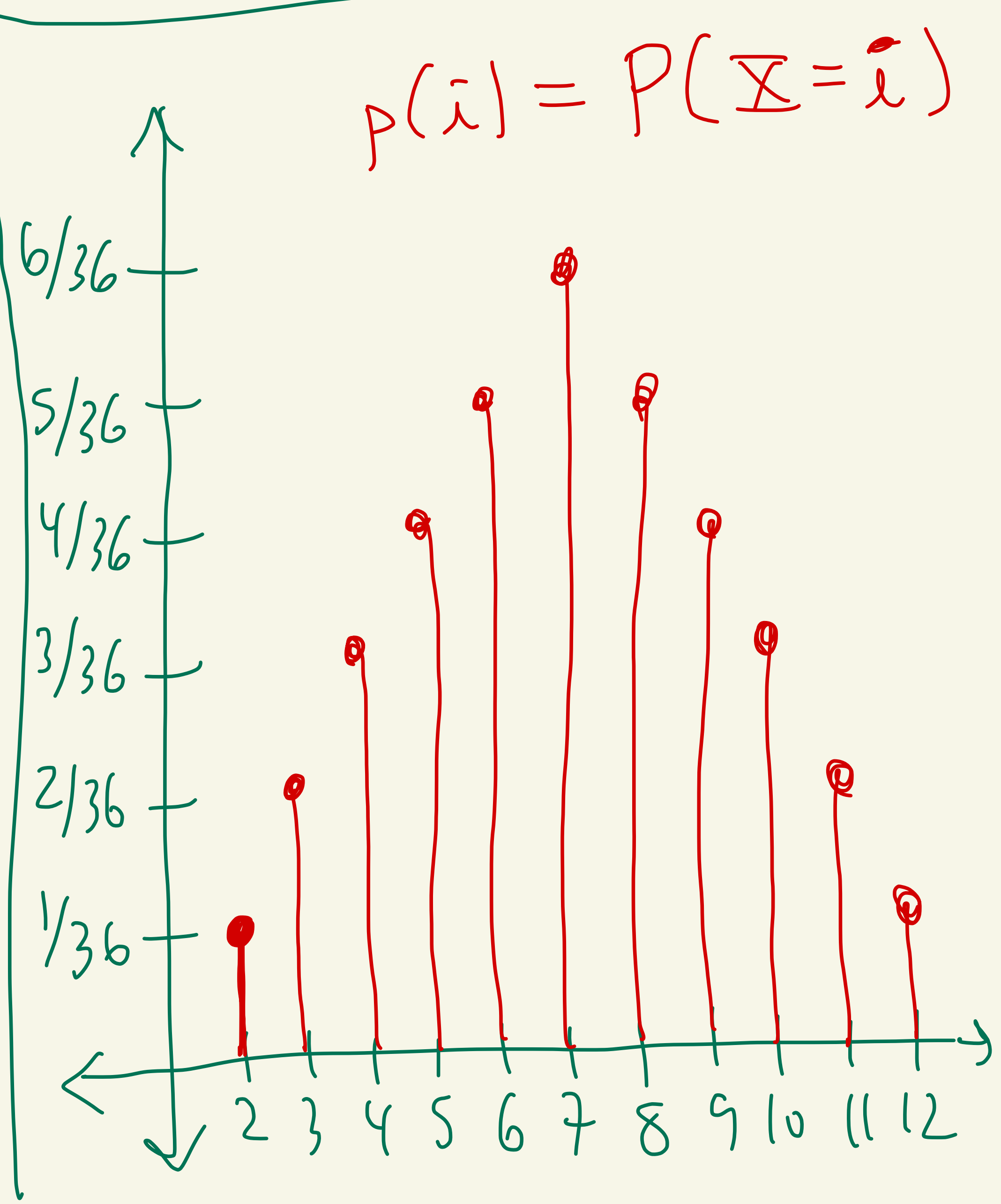
Let's calculate.

$$p(2) = P(X=2) = P(\{(1,1)\}) = 1/36$$

$$p(3) = P(X=3)$$
$$= P(\{(1,2),(2,1)\}) = 2/36$$

$$p(4) = P(X=4)$$
$$= P(\{(1,3),(2,2),(3,1)\}) = 3/36$$

$$p(5) = 4/36$$
$$p(6) = 5/36$$
$$p(7) = 6/36$$
$$p(8) = 5/36$$
$$p(9) = 4/36$$
$$p(10) = 3/36$$
$$p(11) = 2/36$$
$$p(12) = 1/36$$



What about F?

empty set

$$F(1) = P(X \le 1) = P(\phi) = 0$$

$$F(2) = P(X \le 2)$$
$$= P(\{(1,1)\}) = 1/36$$

$$F(3) = P(X \le 3)$$
$$= P(\{(1,1),(1,2),(2,1)\}) = 3/36$$

OR

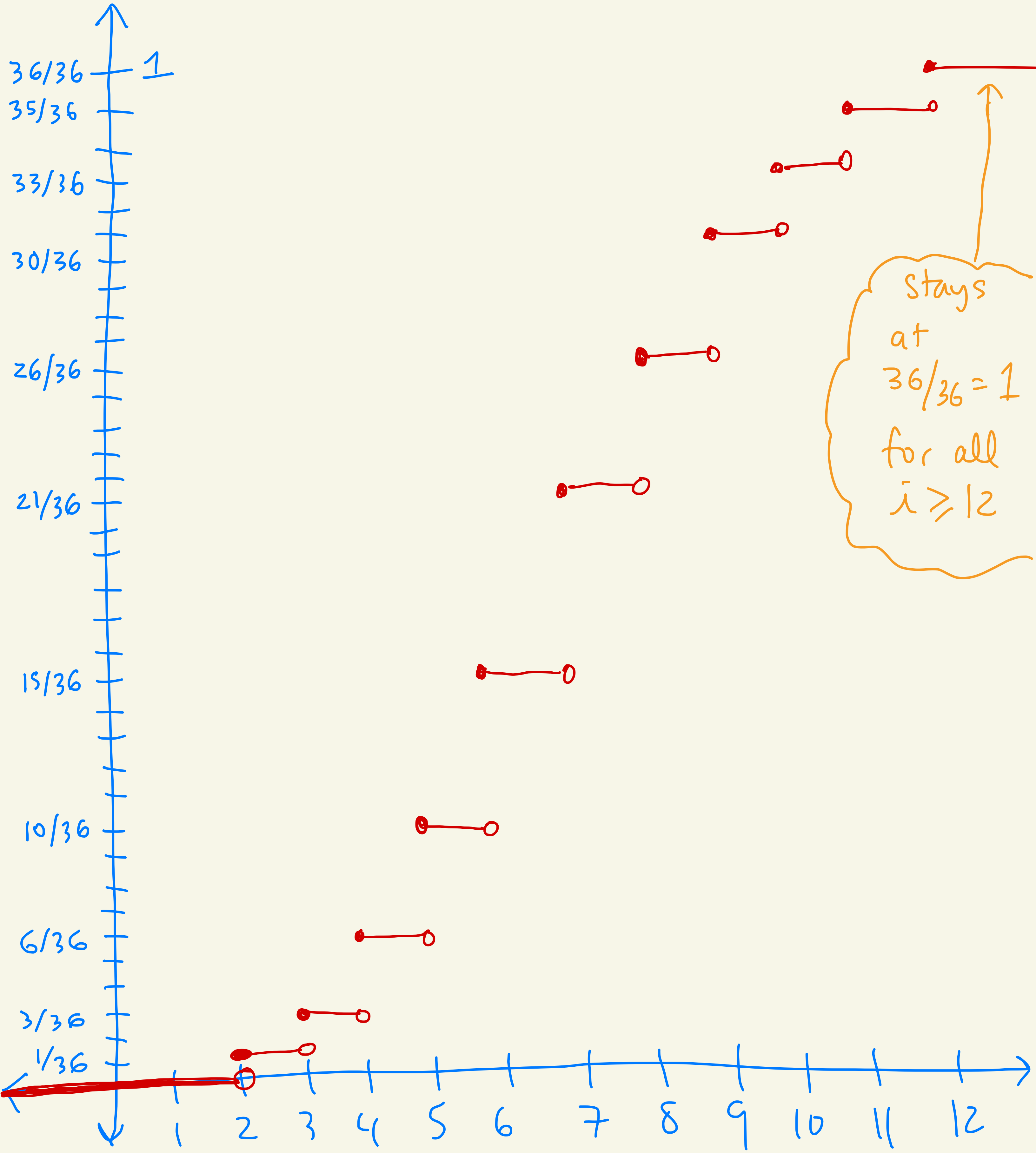
$$F(3) = P(X=2) + P(X=3) = 1/36 + 2/36$$
$$= 3/36$$

$$F(3.7) = P(X \le 3.7)$$
$$= P(X \le 3) = 3/36$$

$$F(4) = P(X \le 4) = P(X=2) + P(X=3) + P(X=4)$$
$$= 1/36 + 2/36 + 3/36 = 6/36$$

and so on...

GRAPH OF $F(i) = P(X \leq i)$



Def: Let X be a discrete random variable on a probability space (S, Ω, P). The expected value of X is

$$\boxed{E[X] = \sum_{\underbrace{\omega \in S}_{\text{sum over outcomes } \omega \text{ in } S}} X(\omega) \cdot P(\{\omega\})}$$

or if the range of X is $x_1, x_2, x_3, \ldots$ then we get

$$\boxed{\begin{aligned} E[X] &= \sum_i x_i \cdot P(X = x_i) \\ &= \sum_i x_i \cdot p(x_i) \end{aligned}}$$

$p(x_i) = P(X = x_i)$

<u>Ex:</u> Let's use the same example as before where we roll two 6-sided die and X is the sum of the dice.

<u>Long way to calculate $E[X]$</u>:

$$E[X] = \sum_{w \in S} X(w) \cdot P(\{w\})$$

$$= \underbrace{X(1,1)}_{2} \cdot \underbrace{P(\{(1,1)\})}_{1/36} + \underbrace{X(1,2)}_{3} \cdot \underbrace{P(\{(1,2)\})}_{1/36}$$

$$+ \underbrace{X(2,1)}_{3} \cdot \underbrace{P(\{(2,1)\})}_{1/36} + \underbrace{X(1,3)}_{4} \cdot \underbrace{P(\{(1,3)\})}_{1/36}$$

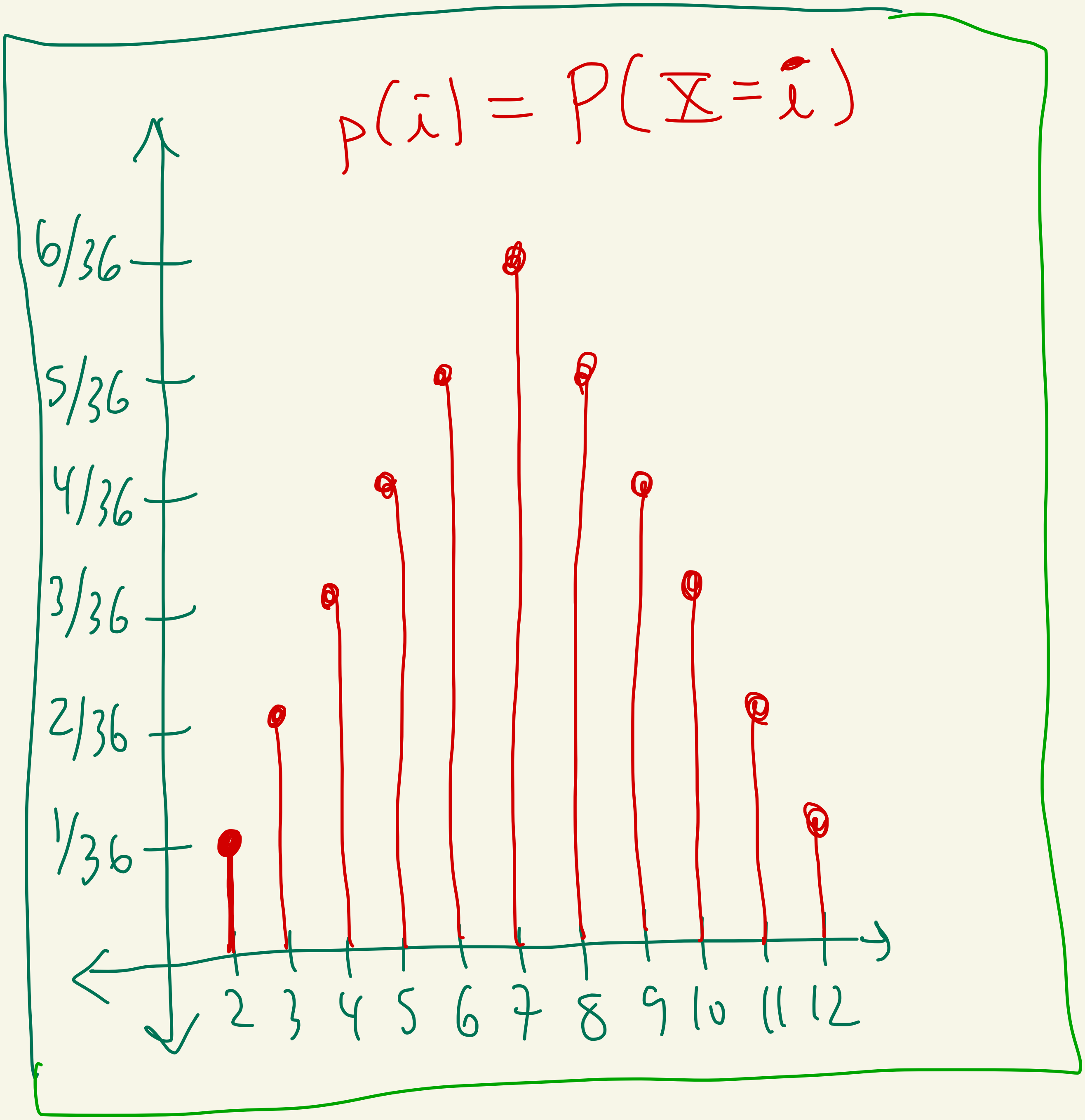
$$+ \underbrace{X(2,2)}_{4} \cdot \underbrace{P(\{(2,2)\})}_{1/36} + \underbrace{X(3,1)}_{4} \cdot \underbrace{P(\{(3,1)\})}_{1/36}$$

$$+ \cdots + \underbrace{X(6,6)}_{12} \cdot \underbrace{P(\{(6,6)\})}_{1/36}$$

$$= 2\left(\frac{1}{36}\right) + (3+3)\left(\frac{1}{36}\right) + (4+4+4)(1/36) + \cdots + (12)(1/36)$$

$$= \frac{1}{36}\Big[2 + 2(3) + 3(4) + 4(5) + 5(6) + 6(7) + 5(8) + 4(9) + 3(10) + 2(11) + 1(12)\Big]$$

$$= 7$$

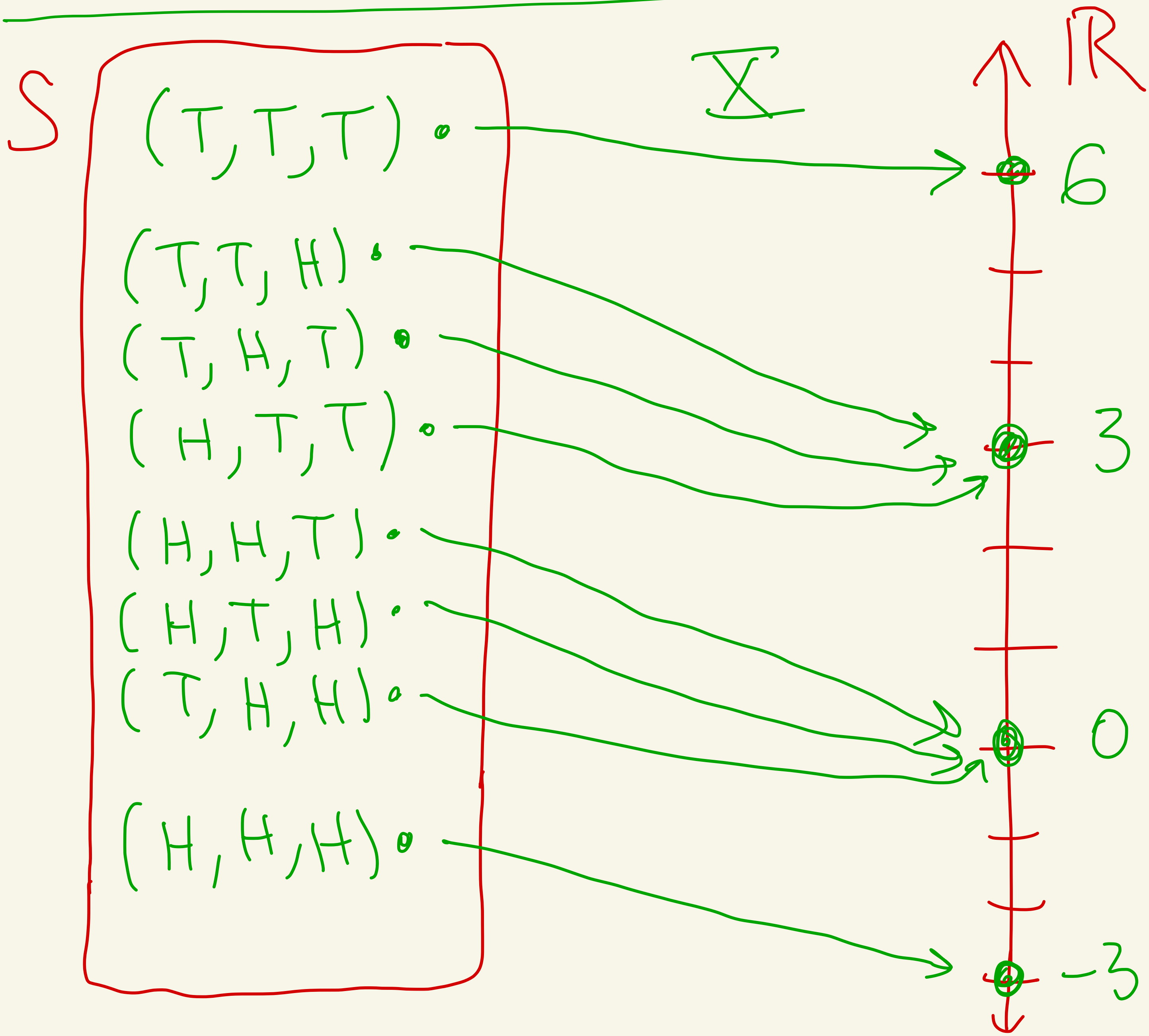


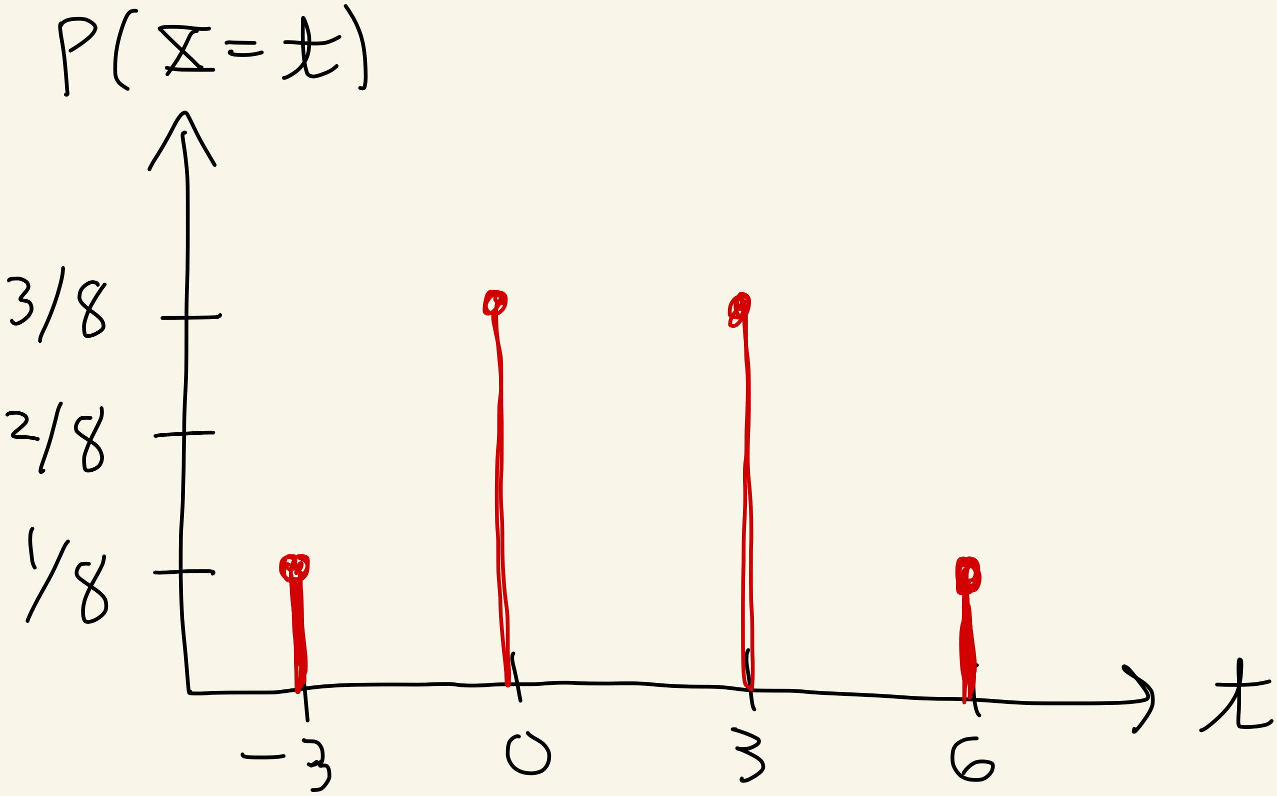
Shorter way to calculate:

$$E[X] = \sum_{t=2}^{12} t \cdot P(X = t)$$

$$
\begin{aligned}
&= (2)\left(\frac{1}{36}\right) + (3)\left(\frac{2}{36}\right) + (4)\left(\frac{3}{36}\right) \\
&\quad + (5)\left(\frac{4}{36}\right) + (6)\left(\frac{5}{36}\right) + (7)\left(\frac{6}{36}\right) \\
&\quad + (8)\left(\frac{5}{36}\right) + (9)\left(\frac{4}{36}\right) + (10)\left(\frac{3}{36}\right) \\
&\quad + (11)\left(\frac{2}{36}\right) + (12)\left(\frac{1}{36}\right) \\
&= 7
\end{aligned}
$$

Ex: Suppose you flip a coin 3 times. For every head you lose \$1. For every tail you win \$2. Let X be the amount won/lost. Draw X and $p(t) = P(X = t)$. Calculate $E[X]$





$$E[X] = (-3)\left(\frac{1}{8}\right) + (0)\left(\frac{3}{8}\right) + (3)\left(\frac{3}{8}\right) + (6)\left(\frac{1}{8}\right)$$

$$= \frac{-3+9+6}{8} = \frac{12}{8} = 1.5$$

This is saying that if you played the game alot of times on average you'd win \$1.50 per play.

So say you played the game 1 million times then you'd expect to win around

$$(1{,}000{,}000)\cdot(\$1.50)$$

$$= \$1{,}500{,}000.$$

Odds Let E be an event.

We define

$$\text{odds for } E = \frac{P(E)}{P(\overline{E})} = \frac{P(E)}{1-P(E)}$$

$$\text{odds against } E = \frac{P(\overline{E})}{P(E)} = \frac{1-P(E)}{P(E)}$$

casino uses this

Ex: Suppose we roll a 4-sided die. Let E be the event that we roll a 1.

So, $P(E) = \frac{1}{4}$.

$$\text{Odds for } E = \frac{P(E)}{1-P(E)} = \frac{1/4}{3/4} = \frac{1}{3}$$ written 1:3 read "1 to 3"

$$\text{odds against } E = \frac{1-P(E)}{P(E)} = \frac{3/4}{1/4} = \frac{3}{1}$$ written 3:1 read "3 to 1"

How to convert odds to probabilities

$\boxed{\text{odds for } E \quad a:b}$ $\longrightarrow$ $\boxed{P(E) = \frac{a}{a+b}}$

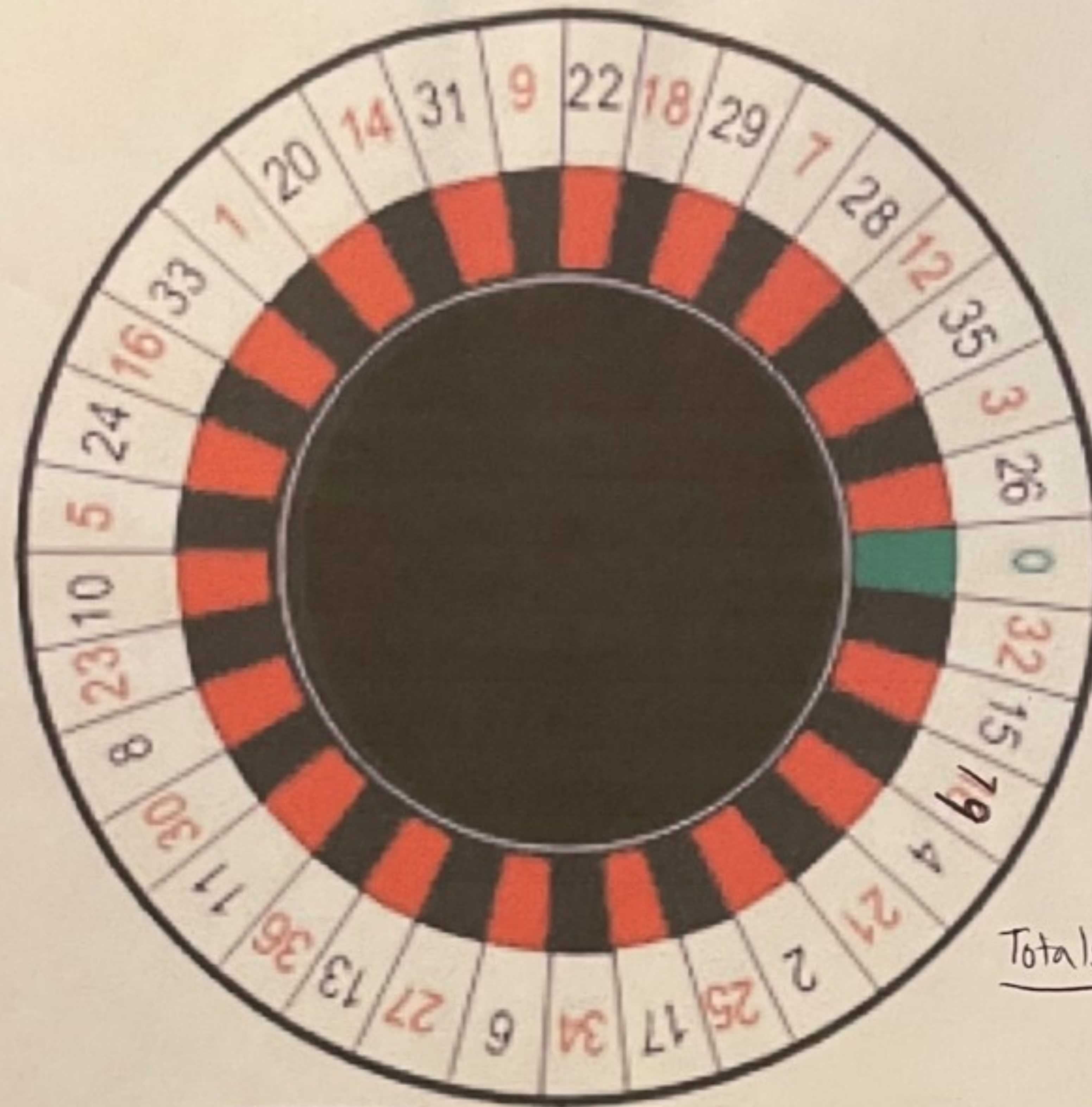
$\boxed{\text{odds against } E \quad c:d}$ $\longrightarrow$ $\boxed{P(E) = \frac{d}{c+d}}$

<u>Ex:</u> Suppose the odds for E are $3:5$. Then $P(E) = \frac{3}{3+5} = \frac{3}{8}$

<u>Ex:</u> Suppose the odds against E are $4:6$. Then $P(E) = \frac{6}{4+6} = \frac{6}{10}$

Let's learn about Roulette.

EUROPEAN



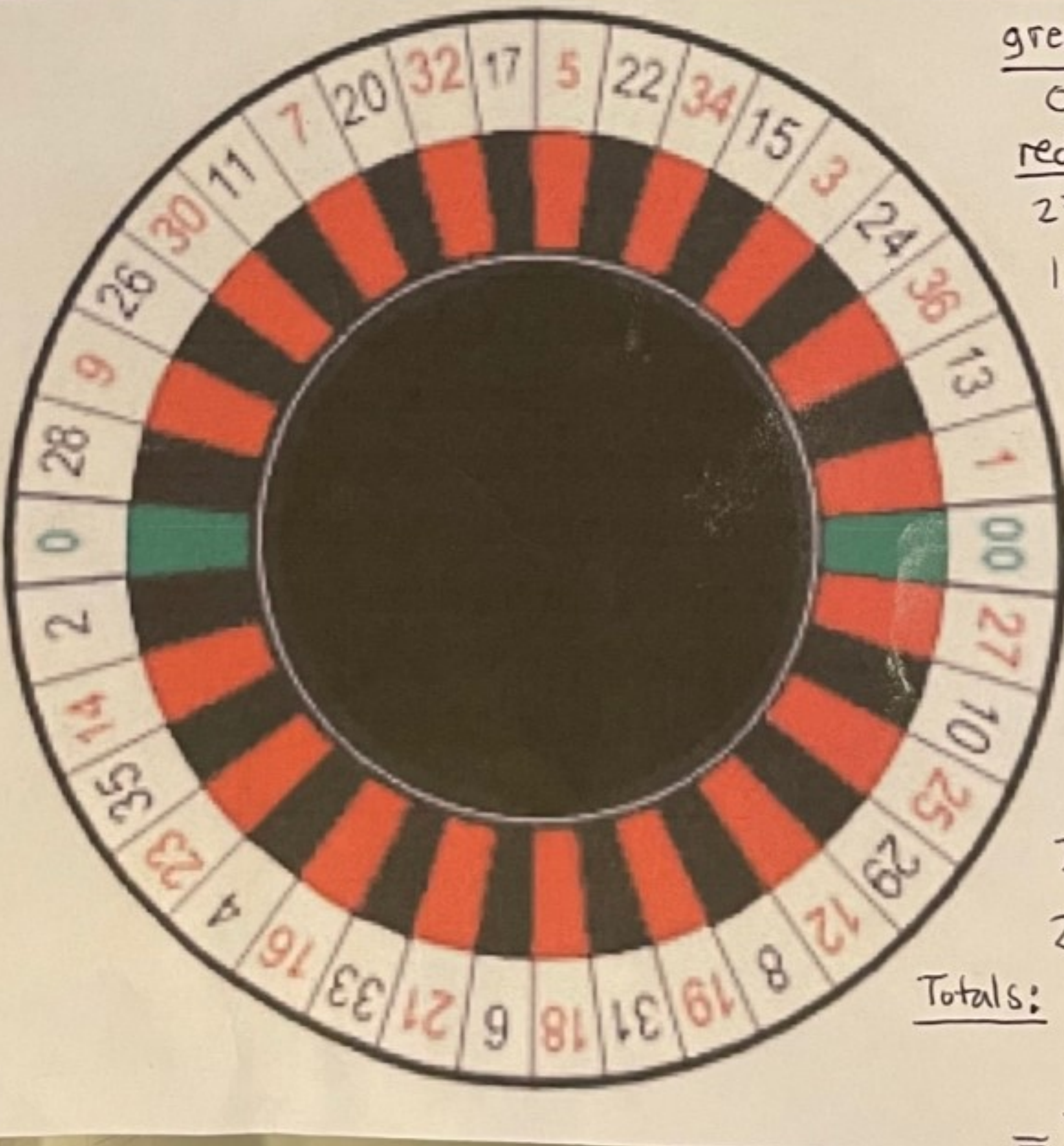
green
0

red
32, 19, 21, 25,
34, 27, 36, 30,
23, 5, 16, 1,
14, 9, 18, 7
12, 3

black
15, 4, 2, 17
6, 13, 11, 8
10, 24, 33, 20
31, 22, 29, 28
35, 26

Totals: 1 green
18 red
18 black
= 37 total

AMERICAN



green
0, 00

red
27, 25, 12, 19
18, 21, 16, 23
14, 9, 30, 7
32, 5, 34, 3
36, 1

black
10, 29, 8, 31
6, 33, 4, 35
2, 28, 26, 11
20, 17, 22, 15
24, 13

Totals: 2 green
18 red
18 black
= 38 total

pg 50

American version / Handout

Casino payouts

Type of Bets And ~~Winning Chances~~

Inside bets

Bet Name	Ex.	Numbers to bet on	Payout	~~Chances~~	True odds
Straight up	A	30	35:1	~~38:1~~	37:1
Split Bet	B	11 or 14	17:1	~~38:2~~	36:2
Street Bet	C	19, 20, 21	11:1	~~38:3~~	35:3
Corner	D	25, 26, 28, 29	8:1	~~38:4~~	34:4
Five Numbers	E	0, 00, 1, 2, 3	6:1	~~38:5~~	
Line Bet	F	4, 5, 6, 7, 8, 9	5:1	~~38:6~~	32:6

Outside Bets

Bet Name	Ex.	Numbers to bet on	Payout	~~Chances~~	True odds
Column	G	Set of column numbers	2:1	~~38:12~~	26:12
Dozen	H	25 through 36	2:1	~~38:12~~	26:12
Red or Black	I	Red numbers	1:1	~~38:18~~	20:18
Even or Odd	J	Odd numbers	1:1	~~38:18~~	20:18
Low or High	K	19 through 36	1:1	~~38:18~~	20:18

Let's analyze some Roulette bets

Sample space for American wheel

$$S = \{0, 00, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$$

Each number is equally likely with probability $\frac{1}{38}$.

$\boxed{\text{Straight up bet}}$ (35:1 payout)

Suppose we bet \$1 on 27.

Let X be the amount won or lost.

$$X(\omega) = \begin{cases} -1 & \text{if } \omega \neq 27 \\ 35 & \text{if } \omega = 27 \end{cases}$$

Then,

$$E[X] = (-1) \cdot P(X = -1) + (35) \cdot P(X = 35)$$

$$= (-1)\left(\frac{37}{38}\right) + (35)\left(\frac{1}{38}\right)$$

$$= \frac{-2}{38} \approx \boxed{-0.0526}$$

So on average we lose 5.26¢ per dollar bet

The casino pays 35:1 on a straight up bet.

What are the real odds (ie the odds against)?

Here the event we are betting on is $E = \{27\}$.

$$\text{odds against } E = \frac{P(\bar{E})}{P(E)} = \frac{37/38}{1/38}$$

$$= \frac{37}{1}$$

The odds against are 37:1.

What's the expected value if the casino paid 37:1?

The new random variable is:

$$Y(w) = \begin{cases} -1 & \text{if } w \neq 27 \\ 37 & \text{if } w = 27 \end{cases}$$

Then,

$$E[Y] = (-1)\overbrace{\left(\frac{37}{38}\right)}^{P(Y=-1)} + (37)\overbrace{\left(\frac{1}{38}\right)}^{P(Y=37)}$$

$$= 0$$

So if the casino paid 37:1 on a straight up bet everyone breaks even in the long run.

$\boxed{\text{Column bet}}$ (2:1 payout)

Suppose we bet \$1 on the third column. So we are betting on the event

$E = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$.

Let X be the amount won or lost.

Then,

$$X(w) = \begin{cases} -1 & \text{if } w \notin E \\ 2 & \text{if } w \in E \end{cases}$$

$w \notin E$ means w is not in E

$w \in E$ means that w is in E

Then,

$$E[X] = (-1)\cdot P(X=-1) + (2)\cdot P(X=2)$$

$$= (-1)\left(\frac{26}{38}\right) + (2)\left(\frac{12}{38}\right)$$

$$= \frac{-2}{38} \approx \boxed{-0.0526}$$

So on average we lose 5.26¢ per dollar bet in the long run.

What are the true odds for E (ie the odds against E)?

It is $\frac{P(\bar{E})}{P(E)} = \frac{26/38}{12/38} = \frac{26}{12} = \frac{13}{6}$

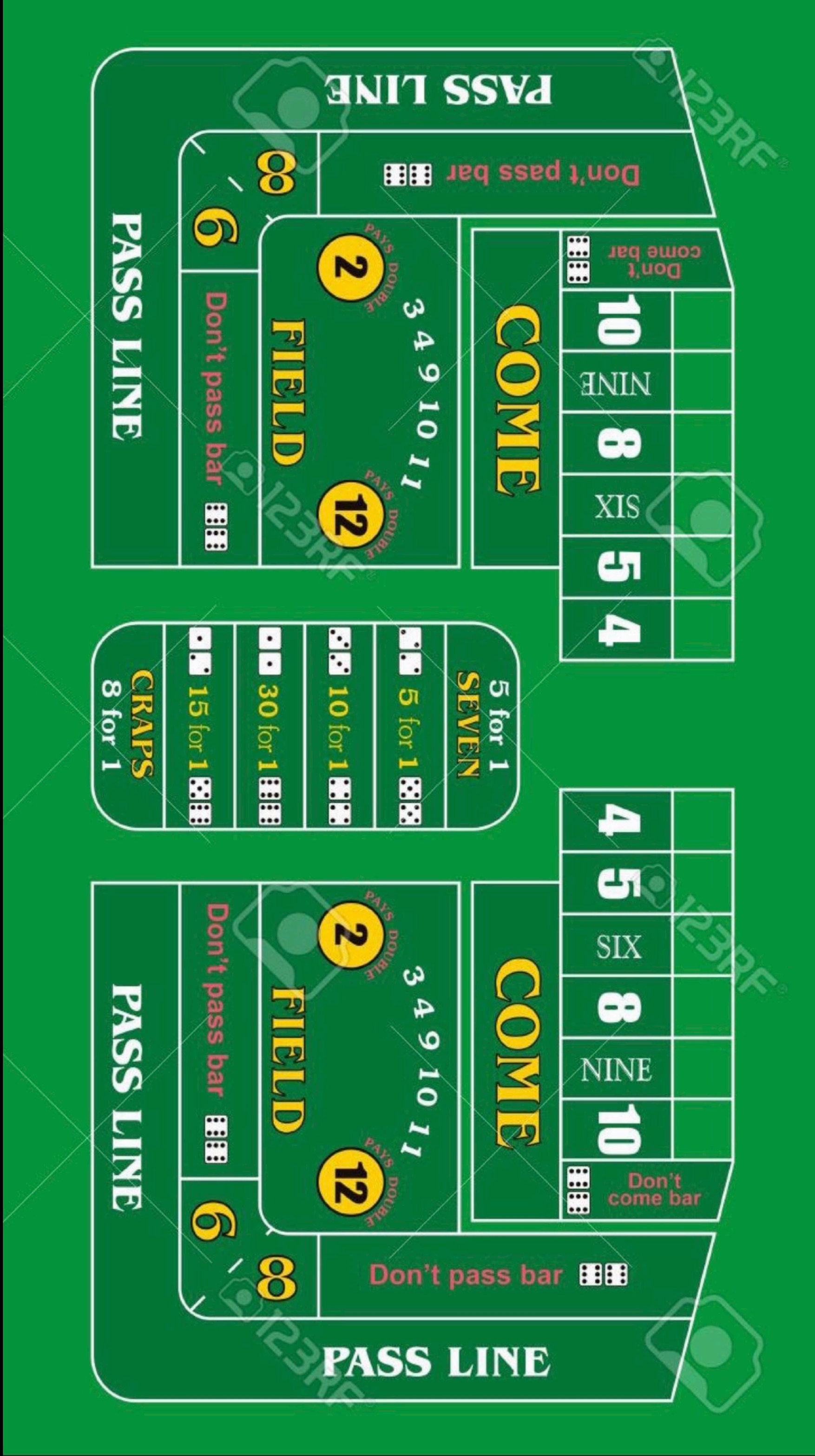
Ie $\boxed{13:6}$

If the casino paid 13:6 on column bets then the expected value would be:

$$(-1)\left(\frac{26}{38}\right) + \left(\frac{13}{6}\right)\left(\frac{12}{38}\right) = 0$$

Then everyone breaks even in the long run.

Let's
learn
about
Craps.

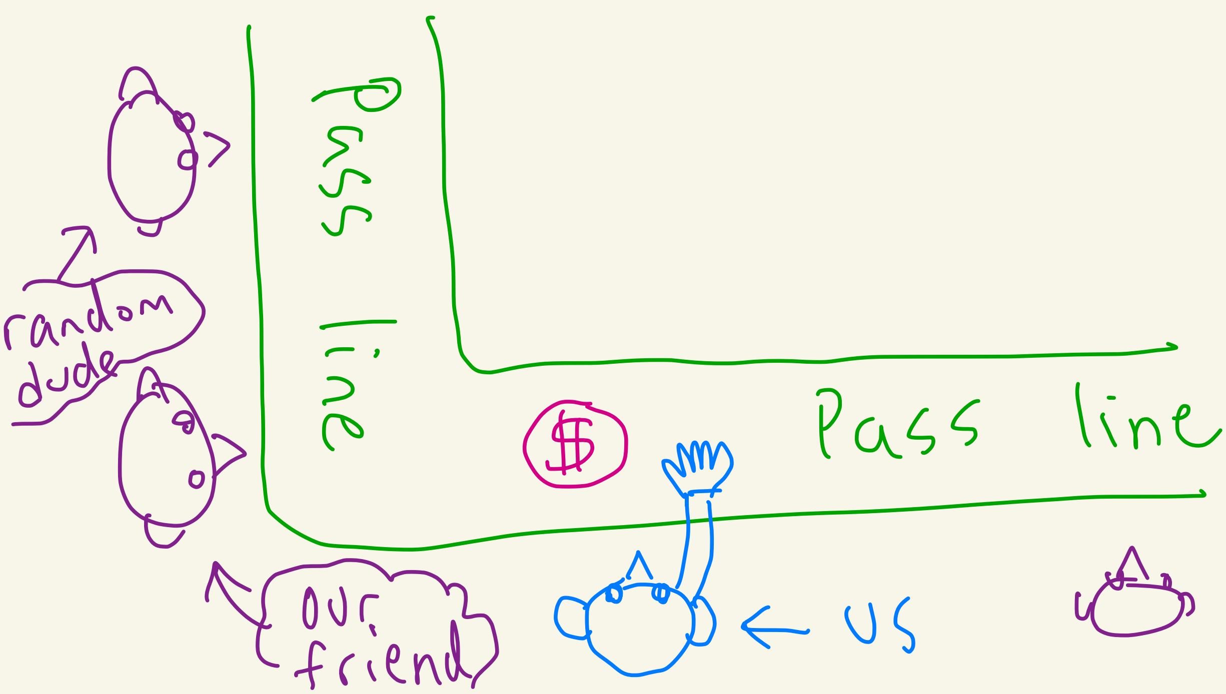

PASS LINE
Don't pass bar
PASS LINE
Don't pass bar
FIELD
2
PAYS DOUBLE
3 4 9 10 11
12
PAYS DOUBLE
COME
Don't come bar
10
NINE
8
SIX
5
4
5 for 1
SEVEN
5 for 1
10 for 1
30 for 1
15 for 1
CRAPS
8 for 1
4
5
SIX
8
NINE
10
Don't come bar
COME
FIELD
2
PAYS DOUBLE
3 4 9 10 11
12
PAYS DOUBLE
Don't pass bar
PASS LINE
6
8
9
Don't pass bar
PASS LINE

Craps

The main bet in craps is called the pass line bet.

People place their bets on the table and the game starts.

Suppose we put money on the pass line.



Some player (called the shooter) rolls the dice.

Two 6-sided dice are rolled.

The first roll is called "the come out roll".

$\boxed{\text{The sum of the dice is measured on each roll.}}$

Case 1: If a 7 or 11 is rolled, then we win the pass line bet.

this roll is called a "natural"

Case 2: If a 2, 3, or 12 is rolled, then we lose the pass line bet.

this roll is called "craps"

Case 3: Suppose a 4, 5, 6, 8, 9, or 10 is rolled. The number rolled is called the "point." Now the dice are rolled over and over again until either 7 is rolled or the point is rolled again.

If 7 comes up first, then we lose the pass line bet.

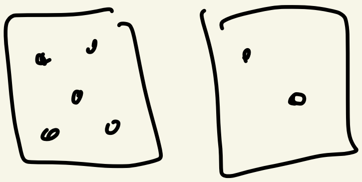
If the point comes up first then we win the pass line bet.

- After one of the 3 cases above occurs, then the game starts over with a new come out roll.
- The casino plays 1:1 on a pass line bet.

Let's see some example games

pass line bet	$2

come out roll



7 is rolled on come out roll

game over

we <u>won</u> $2

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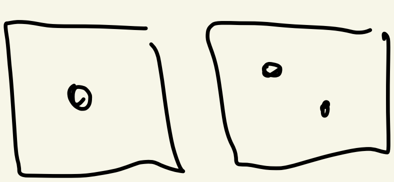
pass line bet	$50

come out roll



3 is rolled on come out roll

game over

we <u>lost</u> $50

pass line bet	$100

come out roll	roll 2	roll 3	roll 4	roll 5
6	2	11	5	6

6 is point
6 is marked on table

point happened before 7

game keeps going until 7 or 6 happens again

We win $100

pass line bet
$10

come out roll	roll 2	roll 3	roll 4	roll 5
5	2	6	6	7

5 is the point

We lose

7 rolled before 5 was rolled again

Let's calculate the expected value of betting on the pass line.

Sum of dice	# ways to roll
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

probabilites for come out roll

roll	probability
7 or 11	8/36
2,3, or 12	4/36
4	3/36
5	4/36
6	5/36
8	5/36
9	4/36
10	3/36

WIN → 7 or 11

LOSE → 2,3, or 12

point is made with probability 24/36

Let's calculate the probabilities of winning or losing once a point is made

Ex: Suppose the come out roll sums to 8. So, 8 is the point.

Now we keep rolling until either an 8 or a 7 comes up.

On an individual roll, let A be the event that the sum is 8 and B be the event that the sum is 7.

If we keep rolling, then...

- the probability that A occurs before B (ie an 8 is rolled before a 7) is

$$\frac{P(A)}{P(A)+P(B)} = \frac{5/36}{5/36+6/36} = \frac{5}{11}$$

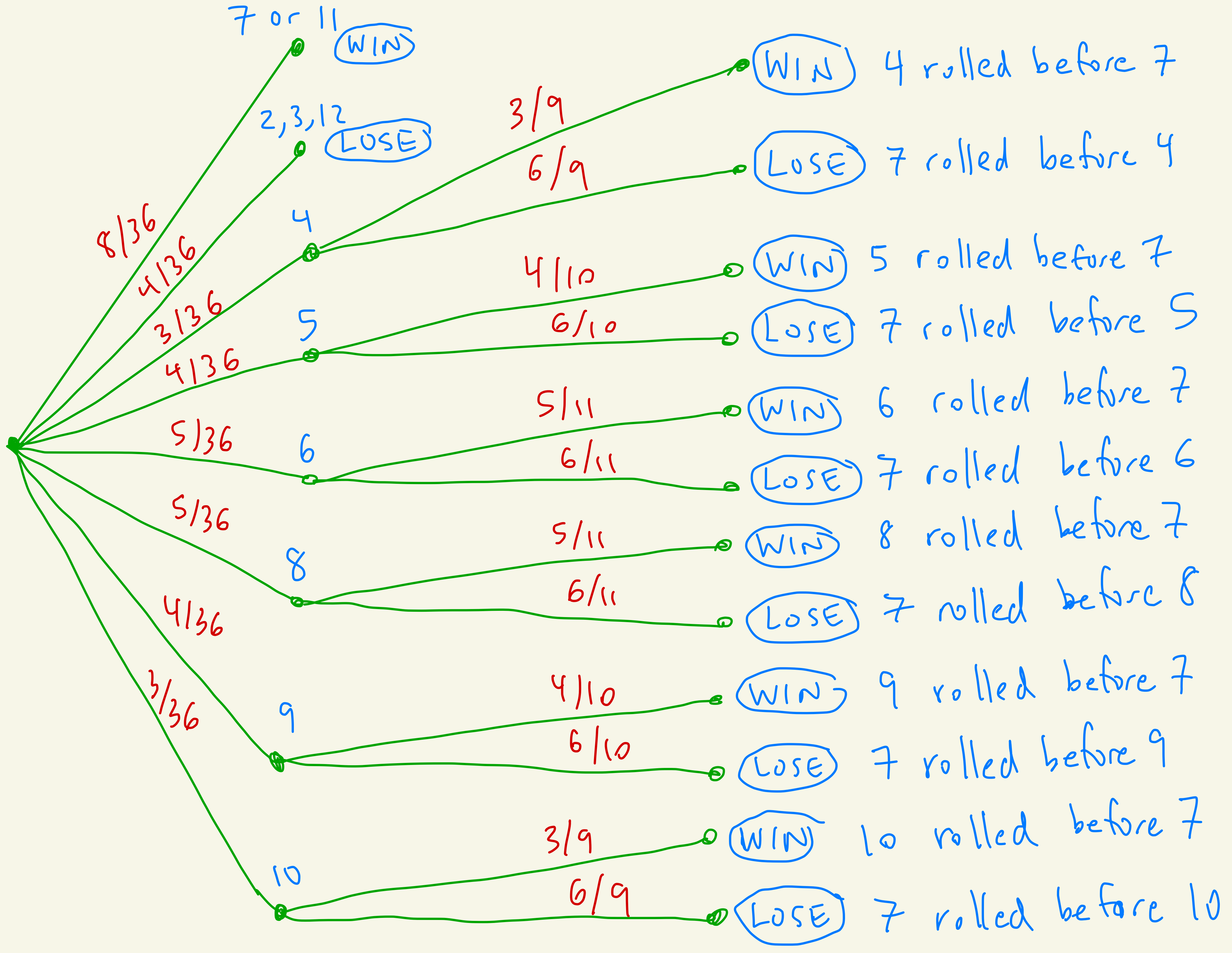
- the probability that B occurs before A (ie a 7 before an 8)

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36+5/36} = \frac{6}{11}$$

Here's the table for all possible points.

point	probability of point being rolled before 7	probability of 7 rolled before point
4	3/9	6/9
5	4/10	6/10
6	5/11	6/11
8	5/11	6/11
9	4/10	6/10
10	3/9	6/9

Let's make the tree of all possibilities.


7 or 11
WIN
2,3,12
LOSE
8/36
4/36
3/36
4/36
5/36
5/36
4/36
3/36
4
3/9
WIN
4 rolled before 7
6/9
LOSE
7 rolled before 4
5
4/10
WIN
5 rolled before 7
6/10
LOSE
7 rolled before 5
6
5/11
WIN
6 rolled before 7
6/11
LOSE
7 rolled before 6
8
5/11
WIN
8 rolled before 7
6/11
LOSE
7 rolled before 8
9
4/10
WIN
9 rolled before 7
6/10
LOSE
7 rolled before 9
10
3/9
WIN
10 rolled before 7
6/9
LOSE
7 rolled before 10

The probability of winning a pass line bet is

$$\frac{8}{36} + \frac{3}{36}\cdot\frac{3}{9} + \frac{4}{36}\cdot\frac{4}{10} + \frac{5}{36}\cdot\frac{5}{11}$$

$$+ \frac{5}{36}\cdot\frac{5}{11} + \frac{4}{36}\cdot\frac{4}{10} + \frac{3}{36}\cdot\frac{3}{9}$$

$$= \boxed{\frac{244}{495}} \approx \boxed{0.4929}$$

The probability of losing a pass line bet is

$$1 - \frac{244}{495} = \boxed{\frac{251}{495}} \approx \boxed{0.5071}$$

Expected value

(Pass line bet is paid 1:1)

Suppose you bet \$1 on the pass line. Let X be the amount won or lost.

$$E[X] = \underbrace{(\$1)\left(\frac{244}{495}\right)}_{\text{WIN}} + \underbrace{(-\$1)\left(\frac{251}{495}\right)}_{\text{LOSE}}$$

$$= \boxed{-\$\frac{7}{495}} \approx \boxed{-\$0.01414\ldots}$$

The 1:1 payout is less than the true odds, ie the odds against winning.

$$\text{True odds} = \frac{p(\text{losing})}{p(\text{winning})}$$

$$= \frac{251/495}{244/495} = \frac{251}{244}.$$

If the casino paid you $\frac{251}{244}:1$ then the expected value would be

$$\underbrace{\left(\$\frac{251}{244}\right)\left(\frac{244}{495}\right)}_{\text{WIN}} + \underbrace{(-\$1)\left(\frac{251}{495}\right)}_{\text{LOSE}} = \$0$$

But then its break even for the casino in the long run.

However, the casino does allow an extra "free odds" bet if a point is made. The free odds bets are paid off at their true odds making them a "fair bet"

[fair bet means expected value 0, ie casino has no edge]

point	true odds
4	2:1
5	3:2
6	6:5
8	6:5
9	3:2
10	2:1

point is 4

$p(\text{win}) = 3/9$

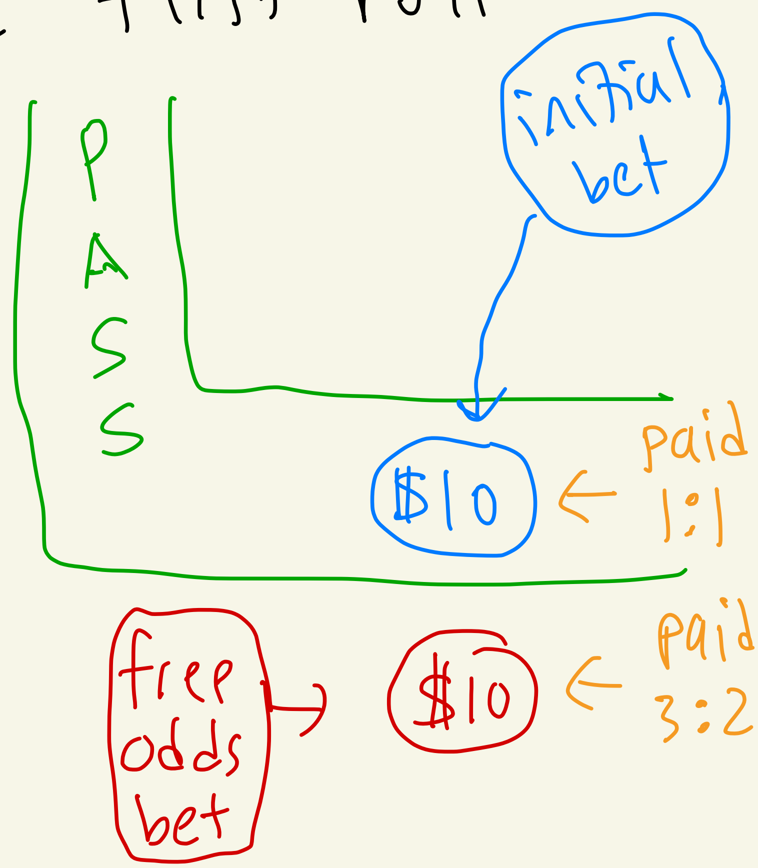
$p(\text{lose}) = 6/9$

$$\text{odds against} = \text{true odds} = \frac{p(\text{lose})}{p(\text{win})}$$

$$= \frac{6/9}{3/9} = \frac{6}{3} = \frac{2}{1}$$

Ex: Suppose you bet \$10 on the pass line. The first roll gives point 5.

Now that the point is made you can make a "free odds" bet. Let's say we bet \$10 more as a free odds bet



come out roll	roll 2	roll 3	roll 4
⚃ ⚀	⚀ ⚁	⚀ ⚂	⚁ ⚂
5	3	4	5

WIN

We would win

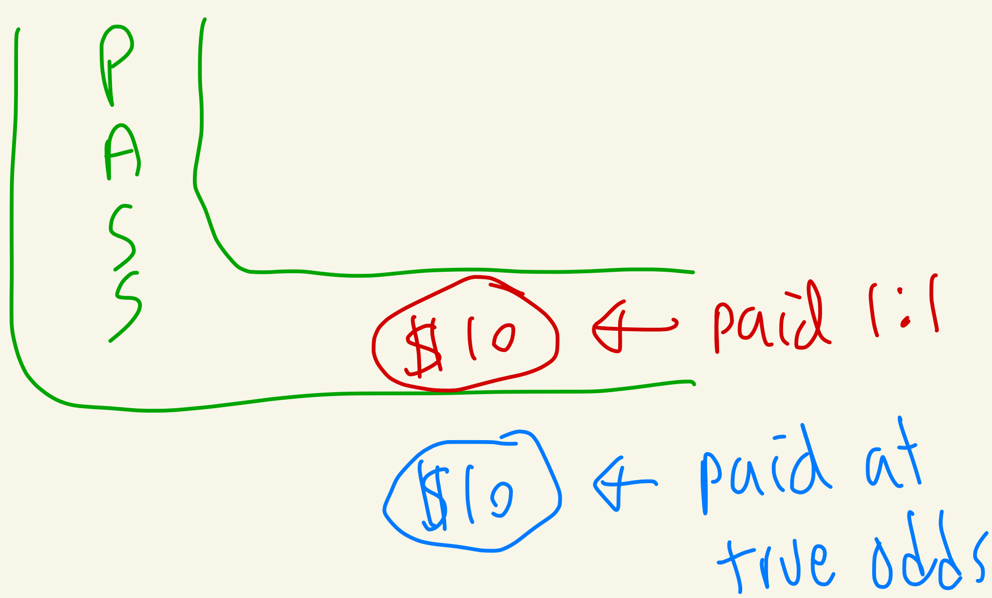
$$\underbrace{(\$10)}_{1:1} + \underbrace{\left(\frac{3}{2}\right)(\$10)}_{\text{free odds}} = \boxed{\$25}$$

If we would have lost we would have lost $\boxed{\$20}$

Let's look at an expected value for free odds betting.

Suppose you bet \$10 on the pass line and if a point is made then you bet an additional \$10 as a free odds bet. Let X be the amount won or lost.

Let's calculate $E[X]$.



$$E[X] = \underbrace{(\$10)\left(\frac{8}{36}\right)}_{\text{7 or 11 on come out roll}} + \underbrace{(-\$10)\left(\frac{4}{36}\right)}_{\text{2, 3, or 12 on come out roll}}$$

$\boxed{\text{4 or 10}}$ $\qquad$ $\boxed{\text{4 or 10}}$

$$+ \underbrace{2\cdot(\$30)\left(\frac{3}{36}\right)\left(\frac{3}{9}\right)}_{\text{4 or 10 as point and we won}} + \underbrace{2\cdot(-\$20)\left(\frac{3}{36}\right)\left(\frac{6}{9}\right)}_{\text{4 or 10 as point and we lost}}$$

\$10 ← pass line 1:1
\$20 ← free odds 2:1
\$30

−\$10 ← pass line
−\$10 ← free odds
−\$20

$\boxed{\text{5 or 9}}$ $\qquad$ $\boxed{\text{5 or 9}}$

$$+ \underbrace{2\cdot(\$25)\left(\frac{4}{36}\right)\left(\frac{4}{10}\right)}_{\text{5 or 9 as point and we won}} + \underbrace{2\cdot(-\$20)\left(\frac{4}{36}\right)\left(\frac{6}{10}\right)}_{\text{5 or 9 as point and we lost}}$$

\$10 ← pass line 1:1

−\$10 ← pass line

$\$15 \leftarrow$ free odds 3:2
$\$25$

$-\$10 \leftarrow$ free odds
$-\$20$

$$+\underbrace{\overset{\boxed{6\text{ or }8}}{2}\cdot(\$22)\left(\frac{5}{36}\right)\left(\frac{5}{11}\right)}_{\text{6 or 8 as point and we won}} + \underbrace{\overset{\boxed{6\text{ or }8}}{2}\cdot(-\$20)\left(\frac{5}{36}\right)\left(\frac{6}{11}\right)}_{\text{6 or 8 as point and we lost}}$$

6 or 8 as point and we won
$\$10 \leftarrow$ pass line 1:1
$\$12 \leftarrow$ free odds 6:5
$\$22$

6 or 8 as point and we lost
$-\$10 \leftarrow$ pass line
$-\$10 \leftarrow$ free odds
$-\$20$

$$= \boxed{-\$\frac{14}{99}} \approx \boxed{-\$0.1414}$$

So if you follow this betting strategy then you lose about 14¢ per game in the long run.

Let's now put the above in "per \$1 bet" terms to compare with our non-free odds betting expected value. Let's see what the average amount bet is with our \$10 pass line / \$10 free odds betting scheme is.

$$\begin{pmatrix}\text{average}\\ \text{amount}\\ \text{bet}\end{pmatrix} = \underbrace{\overbrace{(\$10)}^{\text{amount bet}}\overbrace{\left(\frac{12}{36}\right)}^{\text{probability we bet it}}}_{\text{come out roll is } 7,11,2,3,12} + \underbrace{\overbrace{(\$20)}^{\text{amount bet}}\overbrace{\left(\frac{24}{36}\right)}^{\text{probability we bet it}}}_{\text{come out roll } 4,5,6,8,9,10}$$

$$= (\$10)\left(\frac{1}{3}\right) + (\$20)\left(\frac{2}{3}\right)$$

$$= \left(\$\frac{50}{3}\right) \approx \boxed{\$16.67}$$

Expected value per dollar wagered is

$$\frac{-\$0.1414}{\$16.67} \approx \boxed{-\$0.0085}$$

Recall that with just \$1 bet on the pass line and no free odds bet the expected value was $\boxed{-\$0.01414}$

St. Petersburg Paradox

Goes back to the 1700's.

A casino offers a game to a single player. A fair coin is tossed at each stage.

The pot (amount won) starts at \$2 and doubles every time a head is flipped. The first time a tail is flipped the game ends and the player wins whats in the pot.

How much would you pay to play this game? You don't get back what you paid, just what you win.

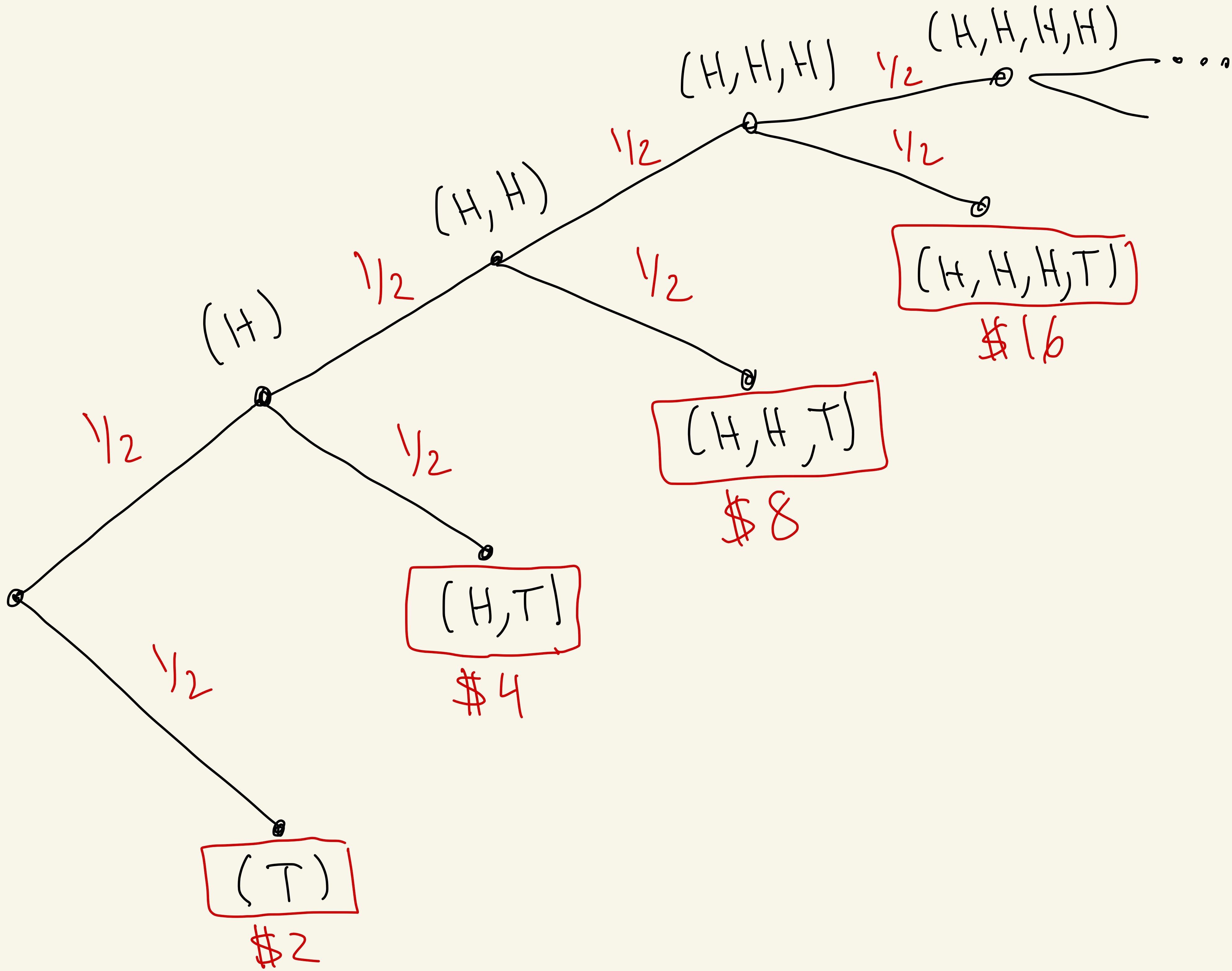
Ex: pay \$5 to play

Pot (amount won)	flip
\$2	H
\$4	H
\$8	H
\$16	T

game stops

winnings = \$16
amount paid = −\$5

$\boxed{\text{net amount won} = \$11}$



(H,H,H,H)
1/2
...
(H,H,H)
1/2
1/2
(H,H,H,T)
$16
(H,H)
1/2
1/2
(H,H,T)
$8
(H)
1/2
1/2
(H,T)
$4
1/2
1/2
(T)
$2

Let X be the net amount won.

$$E[X] = (-\$\text{ amount paid})$$

$$+ (\$2)\underbrace{\left(\frac{1}{2}\right)}_{\text{probability won } \$2} + (\$4)\underbrace{\overbrace{\left(\frac{1}{4}\right)}^{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}}_{\text{probability won } \$4}$$

$$+ (\$8)\underbrace{\left(\frac{1}{8}\right)}_{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} + (\$16)\left(\frac{1}{16}\right) + \dots$$

$$= (-\$\text{ amount paid})$$

$$+ \$1 + \$1 + \$1 + \dots$$

$$= \infty$$

This game has infinite expected value

But winning $\$2^n$ has probability $\frac{1}{2^n}$.

To win at least $\$2^{20} = \$1{,}048{,}576$ has probability equal to

$$\underbrace{\frac{1}{2^{20}}}_{\substack{\text{probability} \\ \text{to win} \\ \$2^{20}}} + \underbrace{\frac{1}{2^{21}}}_{\substack{\text{probability} \\ \text{to win} \\ \$2^{21}}} + \underbrace{\frac{1}{2^{22}}}_{\substack{\text{probability} \\ \text{to win} \\ \$2^{22}}} + \cdots$$

$$= \frac{1}{2^{20}}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2^{20}}\left[\frac{1}{1-1/2}\right] = \frac{1}{2^{20}}\left[2\right] = \frac{1}{2^{19}} \approx$$

$1 + x + x^2 + \cdots = \frac{1}{1-x}$ if $-1 < x < 1$

$x = 1/2$ here geometric sum

$\approx \boxed{0{,}00001907\ldots}$